Parvatibai Chowgule College of Arts and Science
Autonomous
B.Sc. Semester End Examination , January 2022

Semester: III
Subject: Mathematics
Title: Algebra - I
Duration: 2 Hours
Max Marks: 60

Instructions:

- All Questions are compulsory. However, internal choice is applicable.
- Figures to the right indicate full marks.
- Justify all responses.

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1. **Answer ANY THREE of the following:** **(3 X 4 = 12)**

   (a) Show that $\mathbb{R}^2$ is a group under componentwise addition.

   (b) Prove that the intersection of two subgroups is a subgroup.

   (c) If P = $\{\sigma \in S_5$: σ is an odd permutation $\}$. Then find all the possible orders of the elements of P.

   (d) Show that $3\mathbb{Z}$ is prime ideal in $\mathbb{Z}$ while $6\mathbb{Z}$ is not.

   (e) Let $G = \mathbb{Z}_2 \oplus \mathbb{Z}_3$, Find the elements of G and check if G is cyclic.

2. **(A) Answer ANY ONE of the following:** **(6)**

   (a) Let G = $\mathbb{Z}_{32}$, then find all the possible subgroups of G . Further find the generators of the subgroup of order 8 in G.

   (b) What do you mean by the centralizer of element 'a' in a group G? Further find the centralizer of $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ in $GL(2, \mathbb{R})$.

   (B) Suppose that order of $\mid G \mid$ = pq , where p and q are prime. Prove that every proper subgroup of G is cyclic. **(6)**

3. **Answer the following questions.** **(6 + 6 = 12)**

   (a) Let G be a group and a $\in$ G then show that the mapping ϕ_a defined as ϕ_a(x) = axa^{-1} $\forall x \in$ G , is an isomorphism from G to G.

   (b) Let σ= $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{bmatrix}$ and β= $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 4 & 3 & 5 \end{bmatrix}$. Then compute σ^{-1} , $\beta\sigma$, σ β.

(P.T.O.)

4. **Answer the following questions.** **(6 + 6 = 12)**

(a) Let f : $\mathbb{C} \to \mathbb{C}$ be given by $f(a+ib) = (2a - i2b)$ $\forall (a+ib) \in \mathbb{C}$. Then show that f is a ring homomorphism and also find the $ker(\phi)$.

(b) Find the stabilizer and orbit of 3 and 2 in group G where
G ={(1) , (12)(34) , (1234)(56) , (13)(24) , (1432)(56) , (56)(13) , (14)(23) , (24)(56) }

5. **Answer any TWO of the following.** **(2 X 6 = 12)**

(a) Let N be a normal subgroup of G and let H be a subgroup of G. If N is a subroup of H, prove that H/N is a normal subgroup of G/N if and only if H is a normal subgroup of G.

(b) Show that the set of all diagonal matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ such that a , b $\in \mathbb{Z}$, is a subring of the ring of all 2 X 2 matrices over $\mathbb{Z}$.Also describe the elements of $M_2(\mathbb{Z})$ that have multiplicative inverse.

(c) Let R be a commutative ring with unity 1. Prove that
F = {a $\in R : a^{-1}$ exists if a $\neq 0$ } U { 0 , 1} is an integral domain.
Find the characteristic of $\mathbb{Z} \oplus \mathbb{Z}$ and $M_2(3\mathbb{Z})$.

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